Instructions:

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Please do not write under the staple. Make clear what work goes with which problem. Put your name or initials on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils, and your coursepack. You may use any result in the coursepack (whether boxed or an exercise). However, you must cite it, and you may not use it to trivialize an exam question (e.g. to prove itself or a portion of itself or a special case of itself). The first six problems are out of 24 points each (your score will be 12-24 points); the last ten problems are out of 36 points each (your score will be 18-36 points). The maximum possible score $(6\times 24+10\times 36)$ is 504/500 (101%). You have 120 minutes.

Problems out of 24 points:

1. Let $p\in\mathbb{Z}$. Prove that if p is irreducible then p is prime.
(This is half of an exercise. Do not just cite the exercise.)
2. Let $a\in\mathbb{Z}$. Prove that either there is some $k\in\mathbb{Z}$ with $a^4=5k$, or there is some $k\in\mathbb{Z}$ with $a^4=5k+1$.
3. Set $S=\{2a+b\sqrt{-5}:a,b\in\mathbb{Z}\}$. Prove or disprove that this is a subring of $\mathbb{C}$. If yes, determine (with proof) if it is an integral domain.
4. Let $\mathbb{F}$ be a field, and let $f(x),g(x)\in\mathbb{F}[x]$. Prove that the following are equivalent:
(i) $f(x),g(x)$ are associates; (ii) $f(x)|g(x)$ and $g(x)|f(x)$.
5. Consider the ring $\mathbb{R}[x]$, and the subset $S=\{f(x):f(1)=0\}$. Prove or disprove that S is an ideal.
6. Let R be a (commutative) ring with ideal I. We call $x\in R$ *idempotent* if $x^2=x$. Prove that every element of R/I is idempotent, if and only if, $\forall a\in R,\ a^2-a\in I$.

Problems out of 36 points:

7. Prove the uniqueness part of the $\mathbb{Z}$ Division Algorithm Theorem.
8. Use the $\mathbb{Z}$ Euclidean algorithm to find $[25]^{-1}$ in $\mathbb{Z}_{43}$.
9. Recall $M_2(\mathbb{Z})=\left\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right):a,b,c,d\in\mathbb{Z}\right\}$, the ring of 2×2 matrices with integer entries. Consider $S=\left\{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right):a,b,d\in\mathbb{Z}\right\}$. Prove or disprove that S is a subring of $M_2(\mathbb{Z})$. If yes, determine (with proof) if it is an integral domain.
10. Let $\mathbb{F}$ be a field, and let $f(x),g(x)\in\mathbb{F}[x]$, not both zero. Prove that $\gcd(f(x),g(x))$ is unique; that is, prove there cannot be two different monic common divisors of the same (maximal) degree.
11. Prove the Max Root Theorem (from unit 3).
12. Let I be an ideal of $\mathbb{Z}$. Prove that I is principal.
HINT: Two cases: Either I contains at least one positive integer, or not. If not, try to prove that $I=\langle 0\rangle$. If it does, let d be the smallest positive integer it contains. Try to prove that $I=\langle d\rangle$.
13. Prove the Injective Homomorphism Theorem.
14. Prove the Kernel to Ideal Theorem.
15. Let $n\in\mathbb{Z}$ with $n\geq 2$. Set $S=(n)$, a principal ideal. Prove that $\mathbb{Z}/S\cong\mathbb{Z}_n$.
16. Let I be a nontrivial (i.e., $|I|\geq 2$) ideal of $\mathbb{Z}$. Prove that I is maximal, if and only if, I is prime.